

RATIONALITY, UNIVERSAL GENERATION AND THE INTEGRAL HODGE CONJECTURE

MINGMIN SHEN

ABSTRACT. We use the universal generation of algebraic cycles to relate (stable) rationality to the integral Hodge conjecture. We show that the Chow group of 1-cycles on a cubic hypersurface is universally generated by lines. Applications are mainly in cubic hypersurfaces of low dimensions. For example, we show that if a generic cubic fourfold is stably rational then the Beauville–Bogomolov form on its variety of lines, viewed as an integral Hodge class on the self product of its variety of lines, is algebraic. In dimension 3 and 5, we relate stable rationality with the geometry of the associated intermediate Jacobian.

1. INTRODUCTION

An algebraic variety X is *rational* if it contains an open subset that can be identified with an open subset of the projective space of the same dimension. It is called *stably rational* if the product of X and some projective space is rational. The rationality problem is to tell whether a given variety is (stably) rational or not. It is one of the most subtle problems in algebraic geometry.

We work over the field \mathbb{C} of complex numbers unless otherwise stated. The Lüroth theorem and Castelnuovo’s criteria settled the rationality problem in dimensions one and two. One breakthrough in dimension three was made by Clemens–Griffiths [6], where they showed that a smooth cubic threefold is not rational. Other important methods that appeared around the same time include Artin–Mumford [3] and Iskovskikh–Manin [15].

The (stable) rationality problem in dimension three is closely related to the geometry of the intermediate Jacobian.

Theorem 1.1 (Clemens–Griffiths [6] and Voisin [28]). *Let X be a smooth projective variety of dimension three and let $(J^3(X), \Theta)$ be its intermediate Jacobian.*

- (1) *If X is stably rational, then the minimal class $\frac{\Theta^{g-1}}{(g-1)!}$ is algebraic.*
- (2) *If X is rational, then the minimal class $\frac{\Theta^{g-1}}{(g-1)!}$ is algebraic and effective (which is equivalent to that $J^3(X)$ is a Jacobian of curves).*

The integral Hodge conjecture is the statement that every integral Hodge class is an *algebraic class*, namely the cohomology class of some integral algebraic cycle. It is known that the integral Hodge conjecture is false in general. The relation between the rationality problem and the integral Hodge conjecture is very mysterious. Theorem 1.1, especially the first statement, can be viewed as a beautiful connection between (stable) rationality and the integral Hodge conjecture. In this paper, we develop a method to achieve more such statements. The main applications will be given to cubic threefolds and cubic fourfolds. We first recall the definition of a decomposition of the diagonal.

Definition 1.2 ([27, 28]). Let X be a smooth projective variety of dimension d . We say that X admits a *Chow-theoretical decomposition of the diagonal* if

$$\Delta_X = X \times x + Z, \quad \text{in } \text{CH}_d(X \times X),$$

where $x \in X$ is a closed point on X and Z is an algebraic cycle supported on $D \times X$ for some divisor $D \subset X$. We say that X has a *cohomological decomposition of the diagonal* if the above equality holds in $H^{2d}(X \times X, \mathbb{Z})$.

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One important fact is that a stably rational variety always admits a Chow-theoretical (and hence a cohomological) decomposition of the diagonal. Voisin [28] used decomposition of the diagonal to show new examples of three dimensional varieties which are not stably rational. Along the same line of ideas, Totaro [22] showed that a very general hypersurface of degree in a certain range is not stably rational. Further developments include [7, 11, 12, 13, 14, 17]. The smallest possible degree for a hypersurface to be irrational is three. In dimension three, it is not known whether there exists a smooth cubic threefold that is stably rational or that is not stably rational. When it comes the case of smooth cubic fourfolds, we know there exist rational cubic fourfolds [4, 10, 1]. It is expected that a very general cubic fourfold is not rational. However, no single cubic fourfold has been proven irrational.

1.1. Main results. Let $X \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ be a smooth cubic hypersurface of dimension $d \geq 3$ and let $F = F(X)$ be the variety of lines on X . It is known that F is a smooth projective variety of dimension $2d - 4$. Over F we have the universal family

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ p \downarrow & & \\ F & & \end{array}$$

Then we can view $P \times P$ as a correspondence from $F \times F$ to $X \times X$. Let $h \in \mathrm{CH}^1(X)$ be the class of a hyperplane section.

Theorem 1.3. *Assume that X admits a Chow theoretical decomposition of the diagonal.*

(1) *If $d = 3$, then there exists a symmetric 1-cycle $\theta \in \mathrm{CH}_1(F \times F)$ such that*

$$\Delta_X = X \times x + x \times X + \gamma \times h + h \times \gamma + (P \times P)_* \theta, \quad \text{in } \mathrm{CH}^3(X \times X),$$

for some $\gamma \in \mathrm{CH}_1(X)$.

(2) *If $d = 4$, then there exists a symmetric 2-cycle $\theta \in \mathrm{CH}_2(F \times F)$ such that*

$$\Delta_X = X \times x + x \times X + \Sigma + (P \times P)_* \theta, \quad \text{in } \mathrm{CH}^4(X \times X),$$

where $\Sigma \in \mathrm{CH}^2(X) \otimes \mathrm{CH}^2(X)$ is a symmetric decomposable 4-cycle. Moreover if X is very general, then Σ can be chosen to be zero.

When $d = 4$, the variety F is a hyperkähler fourfold that is deformation equivalent to the Hilbert scheme of two points on a $K3$ -surface. The canonical Beauville–Bogomolov bilinear form

$$\mathfrak{B} : \mathrm{H}^2(F, \mathbb{Z}) \times \mathrm{H}^2(F, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

gives rise to an integral Hodge class $q_{\mathfrak{B}} \in \mathrm{H}^{12}(F \times F, \mathbb{Z})$. We have the corresponding statement at the level of cohomology as follows.

Theorem 1.4. *Let X be a smooth cubic hypersurface of dimension $d = 3$ or 4 and let F be the variety of lines on X . Then X admits a cohomological decomposition of the diagonal if and only if there exists a symmetric $(d - 2)$ -cycle $\theta \in \mathrm{CH}_{d-2}(F \times F)$ such that*

$$[\theta] \cdot \hat{\alpha} \otimes \hat{\beta} = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in \mathrm{H}^d(X, \mathbb{Z})_{\mathrm{tr}}$, where $\hat{\alpha} := P^ \alpha$. If $d = 4$ and X is very general, then the above condition is also equivalent to that the Beauville–Bogomolov form $q_{\mathfrak{B}}$ is algebraic.*

In the case of $d = 3$, the above result has the following interesting application.

Corollary 1.5. *Let X be a smooth cubic threefold and let $(J^3(X), \Theta)$ be its intermediate Jacobian. If X admits a decomposition of the diagonal, then the following statements hold.*

- (1) *The minimal class of $J^3(X)$ is algebraic and supported on a divisor of cohomology class 3Θ .*
- (2) *Twice of the minimal class of $J^3(X)$ is represented by a symmetric 1-cycle supported on a theta divisor.*

In the case of cubic fivefolds, our method can also relate rationality with the geometry of the intermediate Jacobian. The price we pay here is that we have to consider the rationality of X and F simultaneously.

Theorem 1.6. *Let X be a smooth cubic fivefold and let F be its variety of lines. If both X and F admits a Chow theoretical decomposition of the diagonal, then the intermediate Jacobian $J^5(X)$ is a direct summand of a Jacobian of curves (without respecting the principal polarizations).*

1.2. Universal generation. The main ingredient in the proof the above results is the universal generation of the Chow group of 1-cycles on a cubic hypersurface by lines. This universal generation works over a general base field. Let Y and Z be smooth projective varieties defined over a field K . A cycle $\gamma \in \text{CH}^r(Z_{K(Y)})$ is universally generating if a spreading $\Gamma \in \text{CH}^r(Y \times Z)$ induces a universally surjective homomorphism $\Gamma_* : \text{CH}_0(Y) \rightarrow \text{CH}^r(Z)$. This means

$$(\Gamma_L)_* : \text{CH}_0(Y_L) \longrightarrow \text{CH}^r(Z_L)$$

is surjective for all field extensions $L \supset K$.

Theorem 1.7. *Let $X \subset \mathbb{P}_K^{d+1}$ be a smooth cubic hypersurface of dimension $d \geq 3$ and let F be its variety of lines. Assume that $\text{CH}_0(F)$ contains an element of degree one. Then the universal line $P \subset F \times X$ restricts to a universally generating 1-cycle $P|_{\eta_F} \in \text{CH}_1(X_{K(F)})$.*

O. Benoist pointed out that the condition that F admits a 0-cycle of degree one can not be removed since the above universal generation fails when X is the universal cubic hypersurface over the generic point of the moduli space.

1.3. Convention and Notation. Let X be a smooth projective variety of dimension d .

- $\text{H}^p(X)$ denotes $\text{H}^p(X, \mathbb{Z})$ modulo torsion.
- If $\alpha_i \in \text{H}^{k_i}(X)$, $1 \leq i \leq r$, such that $\sum k_i = 2d$ then $\alpha_1 \cdot \alpha_2 \cdots \alpha_r$ denotes the intersection number, namely the class $\alpha_1 \cup \cdots \cup \alpha_r$ evaluated against the fundamental class $[X]$. In the special case of the middle cohomology, we write

$$\langle -, - \rangle_X : \text{H}^d(X) \times \text{H}^d(X) \rightarrow \mathbb{Z}, \quad (\alpha, \beta) \mapsto \alpha \cdot \beta.$$

- Let Λ be a Hodge structure, we use $\text{Hdg}(\Lambda)$ to denote the Hodge classes in Λ . In the special case $\Lambda = \text{H}^{2i}(X)$, we use $\text{Hdg}^{2i}(X)$ to denote the Hodge classes in $\text{H}^{2i}(X)$. The transcendental cohomology $\text{H}^p(X)_{\text{tr}}$ is the group of all elements $\alpha \in \text{H}^p(X)$ such that

$$\alpha \cdot \beta = 0, \quad \text{for all } \beta \in \text{Hdg}^{2d-p}(X).$$

We use $\text{H}^{2i}(X)_{\text{alg}} \subseteq \text{Hdg}^{2i}(X)$ to be the subgroup of algebraic classes. The same notation is defined for \mathbb{Z} and \mathbb{Q} coefficients.

- When X is given an ample class $h \in \text{H}^2(X, \mathbb{Z})$, we use $\text{H}^p(X)_{\text{prim}}$ to denote the associated primitive cohomology.
- For any $\alpha \in \text{H}^i(X)$ and $\beta \in \text{H}^j(Y)$ we use $\alpha \otimes \beta$ denote the element in $\text{H}^{i+j}(X \times Y)$ which is obtained via the Künneth decomposition. Similarly, for $\alpha \in \text{CH}^i(X)$ and $\beta \in \text{CH}^j(X)$, we use $\alpha \otimes \beta$ to denote the decomposable cycle $p_1^* \alpha \cdot p_2^* \beta \in \text{CH}^{i+j}(X \times Y)$.
- η_X denotes the generic point of X .

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2. ON A FILTRATION ON COHOMOLOGY

In this section, we study a filtration on cohomology given by Definition 2.1. If a variety X is rational, then X can be obtained from the projective space \mathbb{P}^d by a successive blow ups and blow downs, with centers of dimension at most $d - 2$. Hence the cohomology of X comes from those centers. In this case the filtration corresponds to the dimensions of the centers.

In the first subsection, we give some general facts about the filtration including the behavior under a smooth blow-up (Proposition 2.3). In the second subsection, we treat an important feature of the filtration, namely that some algebraic cycle of X over a function field gives rise to a bilinear pairing on certain piece of the filtration.

Let X be a smooth projective variety of dimension d . We use $H^p(-)$ to denote the integral cohomology $H^p(-, \mathbb{Z})$ modulo torsion.

Definition 2.1. We define an increasing filtration on the middle cohomology $H^d(X, \mathbb{Z})$ by

$$F^i H^d(X, \mathbb{Z}) := \bigcap_{(Y, \Gamma)} \ker\{[\Gamma]^* : H^d(X, \mathbb{Z}) \rightarrow H^*(Y, \mathbb{Z})\}, \quad i \geq 1,$$

where (Y, Γ) runs through all smooth projective varieties Y of dimension at most $d - i$ and all correspondences $\Gamma \in \text{CH}_l(Y \times X)$ with $l \leq d - 1$. We can similarly define the corresponding notion on $H^*(X)$ and $H^*(X, \mathbb{Q})$.

From the above definition, we see that $F^1 H^d(X, \mathbb{Z})$ consists of all $\alpha \in H^d(X, \mathbb{Z})$ such that $f^* \alpha = 0$ in $H^d(Y, \mathbb{Z})$ for all morphisms $f : Y \rightarrow X$ where $\dim Y \leq d - 1$. Let $\alpha \in H^{d,0}(X)$ be the class represented by a global holomorphic d -form, then $f^* \alpha = 0$ for all $f : Y \rightarrow X$ with $\dim Y \leq d - 1$, since $H^{d,0}(Y) = 0$ for dimension reasons. Thus we have

$$H^{d,0}(X) \oplus H^{0,d}(X) \subseteq F^1 H^d(X, \mathbb{Z}) \otimes \mathbb{C}.$$

2.1. The filtration under a blow up. In this section we take X to be a smooth projective variety of dimension d .

Proposition 2.2. Let X' be another smooth projective variety of dimension d and $\Gamma \in \text{CH}_d(X' \times X)$, then $[\Gamma]$ induces

$$[\Gamma]^* : F^k H^d(X, \mathbb{Z}) \longrightarrow F^k H^d(X', \mathbb{Z}).$$

In particular, for any morphism $f : X' \rightarrow X$, the homomorphism f^* on middle cohomology respects the filtration.

Proof. This can be checked directly using composition of correspondences. \square

Proposition 2.3. Let $\rho : \tilde{X} \rightarrow X$ be a blow up along a smooth center $Y \subset X$ of codimension r . Then

$$\rho^* : F^k H^d(X, \mathbb{Z}) \rightarrow F^k H^d(\tilde{X}, \mathbb{Z})$$

is an isomorphism for all $k \leq r$. In particular, the groups $F^1 H^d(X, \mathbb{Z})$ and $F^2 H^d(X, \mathbb{Z})$ are birational invariants and they vanish if X is rational.

Proof. Consider the blow up diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{X} \\ \pi \downarrow & & \downarrow \rho \\ Y & \xrightarrow{i} & X \end{array}$$

where $E \cong \mathbb{P}(\mathcal{N}_{Y/X})$ is the exceptional divisor. Then it is known that

$$(1) \quad H^d(X, \mathbb{Z}) \oplus \left(\bigoplus_{l=1}^{r-1} H^{d-2l}(Y, \mathbb{Z}) \right) \cong H^d(\tilde{X}, \mathbb{Z}), \quad (\alpha, \beta_1, \dots, \beta_{r-1}) \mapsto \rho^* \alpha + \sum_{l=1}^{r-1} j_* (\xi^{l-1} \cup \pi^* \beta_l),$$

where $\xi \in H^2(E, \mathbb{Z})$ is the class of the relative $\mathcal{O}(1)$ -bundle on E . Since $\rho_* \rho^* = \text{id}$, we have

$$(2) \quad \rho^* : F^k H^d(X, \mathbb{Z}) \hookrightarrow F^k H^d(\tilde{X}, \mathbb{Z}).$$

Now let

$$\tilde{\alpha} = \rho^* \alpha + \sum_{l=1}^{r-1} j_* (\xi^{l-1} \cup \pi^* \beta_l)$$

be an element in $F^k H^d(\tilde{X}, \mathbb{Z})$. Then let

$$\Gamma_{k'} := (\pi, j)_* \xi^{k'-1} \in \text{CH}_{d-k'}(Y \times \tilde{X}), \quad 1 \leq k' \leq r-1,$$

where $(\pi, j) : E \rightarrow Y \times X$ is the natural morphism. Since $\dim Y = d - r \leq d - k$, we have

$$0 = \Gamma_1^* \tilde{\alpha} = \pi_* j^* \tilde{\alpha} = -\beta_{r-1}.$$

Then we apply Γ_2^* to $\tilde{\alpha}$ and get $\beta_{r-2} = 0$. By induction, we get $\beta_l = 0$ for all $l = 1, \dots, r-1$. To show that $\alpha \in F^k H^d(X, \mathbb{Z})$, pick an arbitrary variety Z of dimension $d-k$ and a correspondence $\Gamma \in \text{CH}_{d-k'}(Z \times X)$, $k' \geq 1$. Then

$$\Gamma^* \alpha = \Gamma^* \rho_* \tilde{\alpha} = ({}^t \rho \circ \Gamma)^* \tilde{\alpha} = 0$$

and hence $\alpha \in F^k H^d(X, \mathbb{Z})$. It follows that ρ^* in equation (2) is also surjective. \square

Proposition 2.4. *Let X be a smooth cubic hypersurface of dimension $d = 3$ or 4 . Then*

$$F^1 H^d(X, \mathbb{Z}) = F^2 H^d(X, \mathbb{Z}) = 0, \quad F^3 H^d(X, \mathbb{Z}) = H^d(X, \mathbb{Z})_{\text{tr}}.$$

Proof. We first note that the middle cohomology of X is torsion free. Let F be the variety of lines on X . Let $l \subset X$ be a general line on X and let $D_l \subseteq F$ be the variety of lines that meet l . Then we know that D_l is smooth and the associated Abel–Jacobi homomorphism

$$\Phi_l : H^d(X, \mathbb{Z}) \rightarrow H^{d-2}(D_l, \mathbb{Z})$$

is injective. This can be seen from the intersection property that

$$\Phi_l(\alpha) \cdot \Phi_l(\beta) = [D_l] \cdot \hat{\alpha} \cdot \hat{\beta} = -2\langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in H^d(X, \mathbb{Z})_{\text{prim}}$, see [6, 19]. As a consequence, we see that

$$F^2 H^d(X, \mathbb{Z}) = F^1 H^d(X, \mathbb{Z}) = 0.$$

Let $\alpha \in F^3 H^d(X, \mathbb{Z})$. If $d = 3$, then for all subvarieties $\Sigma \subset X$ of dimension at most 2, we have

$$[\Sigma] \cdot \alpha = 0,$$

which exactly means that $\alpha \in H^3(X, \mathbb{Z})_{\text{tr}} = H^3(X, \mathbb{Z})$. If $d = 4$, then for all curves C and correspondences $\Sigma \in \text{CH}_l(C \times X)$, $l \leq 3$, we have $\Sigma^* \alpha = 0$ in $H^{6-2l}(C, \mathbb{Z})$. This condition is nontrivial only when $l = 2$ or 3 . In either case, it is equivalent to $[Z] \cdot \alpha = 0$ for some surface $Z \subset X$. So we again have $F^3 H^4(X, \mathbb{Z}) = H^4(X, \mathbb{Z})_{\text{tr}}$. \square

2.2. Bilinear form associated to a cycle over a function field.

Definition–Lemma 2.5. *Let K be the function field of a variety of dimension $d - 2r$. For any $\gamma \in \text{CH}_r(X_K)$ we can define a $(-1)^d$ -symmetric bilinear form*

$$\langle -, - \rangle_\gamma : F^{2r+1} H^d(X) \times F^{2r+1} H^d(X) \rightarrow \mathbb{Z}$$

by $(\alpha, \beta) \mapsto \langle \Gamma^* \alpha, \Gamma^* \beta \rangle_Z$, where Z is a model of K and $\Gamma \subset Z \times X$ is a spreading of γ .

Proof. The above bilinear form is independent of the choices made. Indeed, a different choice of Z and Γ gives rise to an action on $H^d(X)$ that differs by the action of a correspondence that factors through a variety of dimension at most $d - 2r - 1$. Hence the difference action is zero on $F^{2r+1} H^d(X)$ by definition. \square

Proposition 2.6. *If $\gamma_1, \gamma_2 \in \text{CH}_r(X_K)$ such that $\gamma_1 - \gamma_2$ is torsion in $\text{CH}_r(X_K)$, then $\langle -, - \rangle_{\gamma_1} = \langle -, - \rangle_{\gamma_2}$.*

Proof. By definition $n(\gamma_1 - \gamma_2) = 0$ in $\text{CH}_r(X_K)$. If we take some spreading $\Gamma_i \in \text{CH}_{d-r}(Z \times X)$ of γ_i , we see that $n(\Gamma_1 - \Gamma_2)$ is supported over a proper closed subset of Z . Hence

$$[n(\Gamma_1 - \Gamma_2)]^* \alpha = 0, \quad \text{for all } \alpha \in F^{2r+1} H^d(X).$$

It follows that $\Gamma_1^* \alpha = \Gamma_2^* \alpha$. \square

3. UNIVERSAL GENERATION

In this section we give the definition of universal generation of algebraic cycles and discuss its basic properties. Then we discuss how universal generation is related to the decomposition of the diagonal.

3.1. Definitions and basic properties. Let X be a smooth projective variety of dimension d over a field k . Let Z be a smooth projective variety with function field $K = k(Z)$. For any cycle $\gamma \in \text{CH}_r(X_K)$, we can define

$$(3) \quad \gamma_* : \text{CH}_0(Z) \rightarrow \text{CH}_r(X), \quad \tau \mapsto \Gamma_* \tau,$$

where $\Gamma \in \text{CH}_{d_Z+r}(Z \times X)$ is a spreading of γ . Namely, $\Gamma|_{\eta_Z \times X} = \gamma$ in $\text{CH}_r(X_K)$. If Γ' is another spreading of γ , then $\Gamma - \Gamma'$ is supported on $D \times X$ for some divisor D of Z . It follows that $\Gamma_* \tau = \Gamma'_* \tau$ for all $\tau \in \text{CH}_0(Z)$. Thus the homomorphism (3) only depends on the class γ . Furthermore, for every field extension $L \supset k$, we have the induced homomorphism

$$(4) \quad (\gamma_L)_* : \text{CH}_0(Z_L) \longrightarrow \text{CH}_r(X_L), \quad \tau \mapsto (\Gamma_L)_* \tau.$$

The above construction can be generalized to the following situation. Let Z be the disjoint union of smooth projective varieties Z_i with function field K_i and $\gamma = \sum_i \gamma_i$, where $\gamma_i \in \text{CH}_r(X_{K_i})$. Then we can again define

$$(5) \quad (\gamma_L)_* : \bigoplus_i \text{CH}_0(Z_i \otimes L) \longrightarrow \text{CH}_r(X_L).$$

Definition 3.1. The cycle

$$\gamma = \sum_{i=1}^n \gamma_i \in \bigoplus_{i=1}^n \text{CH}_r(X_{K_i}),$$

is *universally generating* if the natural homomorphism (5) is surjective for all field extensions $L \supset k$.

Some times we consider universal generation modulo certain subgroup. To explain what those subgroups are, we need the following definitions.

Definition 3.2. Let $L \supset k$ be some function field and let $\gamma \in \text{CH}_r(X_L)$. Then the *variation dimension* of γ , denoted $\text{var}(\gamma)$, is the smallest non-negative integer l for which the following condition holds: there exist function fields K_i of dimension at most l , such that $\gamma = (\gamma'_L)_* \tau$ for some $\gamma' \in \bigoplus_i \text{CH}_r(X_{K_i})$ and some $\tau \in \bigoplus_i \text{CH}_0(Z_i \otimes L)$, where Z_i is a smooth projective model of K_i .

Remark 3.3. The elements of variation dimension 0 form the subgroup $\text{CH}_r(X) \subseteq \text{CH}_r(X_L)$. The above definition can be spelled out in the following way. Let Y be a smooth projective model of L . If the variation dimension of $\gamma \in \text{CH}_r(X_L)$ is l , then we have a smooth projective variety $Z = \coprod_i Z_i$ of dimension l (the largest dimension of its components) and a pull-back diagram

$$\begin{array}{ccccccc} \gamma & \longrightarrow & \Gamma & \longrightarrow & \Gamma' & \longrightarrow & X \\ & & \downarrow & & \downarrow & & \\ & & \tilde{Y} & \longrightarrow & Z & & \\ & & \downarrow & & & & \\ \eta_Y & \longrightarrow & Y & & & & \end{array}$$

This means that $\Gamma = \Gamma' \circ \tilde{Y}$ as correspondences and γ is the restriction of Γ to the generic point of Y . Such diagram no longer exist if we require $\dim Z < l$.

Lemma 3.4. Let $\gamma, \gamma' \in \text{CH}_r(X_L)$. Then

$$\text{var}(\gamma + \gamma') \leq \max\{\text{var}(\gamma), \text{var}(\gamma')\}.$$

Proof. If Z and Z' can be found for γ and γ' respectively. Then we can take the disjoint union of Z and Z' for $\gamma + \gamma'$. \square

Definition 3.5. For any function field L , we define an increasing filtration on $\text{CH}_r(X_L)$ by

$$(6) \quad F_{\text{var}}^p \text{CH}_r(X_L) := \{\gamma \in \text{CH}_r(X_L) : \text{var}(\gamma) \leq p\}.$$

The *variation dimension* of the Chow group $\text{CH}_r(X)$ is defined to be

$$\text{v.dim}(\text{CH}_r(X)) := \sup\{\text{var}(\gamma) : \gamma \in \text{CH}_r(X_L)\},$$

where $L \supset k$ runs through all function fields. The group $\mathrm{CH}_r(X)$ is *universally trivial* if

$$\mathrm{v.dim}(\mathrm{CH}_r(X)) = 0,$$

or equivalently $\mathrm{CH}_r(X) = \mathrm{CH}_r(X_L)$ for all function fields $L \supset k$.

By Lemma 3.4, we see that $F_{\mathrm{var}}^p \mathrm{CH}_r(X_L)$ is a subgroup. We also see that $F_{\mathrm{var}}^p \mathrm{CH}_r(X_L) = \mathrm{CH}_r(X_L)$ whenever $p \geq \mathrm{v.dim}(\mathrm{CH}_r(X))$. The group $\mathrm{CH}_0(X)$ is universally trivial if and only if X admits a Chow-theoretical decomposition of the diagonal; see [29].

Proposition 3.6. *Let X/k be a smooth projective variety of dimension d . The the following are true.*

- (i) *If $\mathrm{CH}_r(X)$ is universally generated by cycles over function fields of dimension at most l , then the variation dimension of $\mathrm{CH}_r(X)$ is less than or equal to l .*
- (ii) *Assume that $k = \mathbb{C}$. Let Z be a smooth projective variety of dimension $d - 2r$ and K be its function field. Let $\gamma_1, \gamma_2 \in \mathrm{CH}_r(X_K)$. If $\mathrm{var}(\gamma_1 - \gamma_2) < d - 2r$, then*

$$\langle \alpha, \beta \rangle_{\gamma_1} = \langle \alpha, \beta \rangle_{\gamma_2}$$

for all $\alpha, \beta \in F^{2r+1}H^d(X, \mathbb{Z})$.

Proof. Statement (i) follows from the definitions. We prove (ii). Let Γ_1 and Γ_2 be spreadings of γ_1 and γ_2 . By the definition variation dimension, we know that $\Gamma_1 - \Gamma_2$ factors through varieties of dimension at most $d - 2r - 1$. It follows that

$$(\Gamma_1 - \Gamma_2)^* \alpha = 0$$

for all $\alpha \in F^{2r+1}H^d(X, \mathbb{Z})$. □

Definition 3.7. We say that the cycle $\gamma \in \bigoplus_i \mathrm{CH}_r(X_{K_i})$ generates $\mathrm{CH}_r(X)$ universally modulo l -dimensional variation if for all function fields $L \supset k$, the homomorphism (5) is surjective modulo $F_{\mathrm{var}}^l \mathrm{CH}_r(X_L)$. In other words, for every $\tau \in \mathrm{CH}_r(X_L)$, there exists $\tau' \in \bigoplus_i \mathrm{CH}_0(Z_i \otimes L)$ such that $\tau - (\gamma_L)_* \tau'$ is in $F_{\mathrm{var}}^l \mathrm{CH}_r(X_L)$.

3.2. Universal generation from decomposition of diagonal.

Proposition 3.8 (Voisin [29]). *Let X be a smooth projective variety of dimension d over \mathbb{C} . Then X admits a Chow-theoretical decomposition of the diagonal if and only if the following condition (*) holds.*

(*) *There exist smooth projective varieties Z_i of dimension $d - 2$, correspondences $\Gamma_i \in \mathrm{CH}_{d-1}(Z_i \times X)$ and integers $n_i \in \mathbb{Z}$, $i = 1, 2, \dots, r$, such that*

$$\Delta_X = \sum_{i=1}^r n_i \Gamma_i \circ {}^t \Gamma_i + X \times x + x \times X, \quad \text{in } \mathrm{CH}_d(X \times X),$$

where $x \in X$ is a closed point.

Proof. The proof is the same as that of [29, Theorem 3.1], with homological equivalence replaced by rational equivalence. We only sketch the main steps here. It is clear that the condition (*) is a special form of Chow-theoretical decomposition of the diagonal. For the converse we assume that X has a Chow-theoretical decomposition of the diagonal

$$\Delta_X - X \times x = Z, \quad \text{in } \mathrm{CH}_d(X \times X),$$

where Z is supported on $D \times X$ for some divisor $D \subset X$. We may relace X by a blow-up and assume that $D = \cup D_i$ is a global normal crossing divisor. Let $k_i : D_i \rightarrow X$ be the inclusion map. Then there exist $\Gamma'_i \in \mathrm{CH}_d(D_i \times X)$ such that

$$\Delta_X - X \times x = \sum (k_i, \mathrm{Id}_X)_* \Gamma'_i = \sum \Gamma'_i \circ k_i^*.$$

Composing the above equation with its transpose, we get

$$\begin{aligned} \Delta_X - X \times x - x \times X &= (\Delta_X - X \times x) \circ (\Delta_X - x \times X) \\ &= \sum_{i,j} \Gamma'_i \circ k_i^* \circ k_{j,*} \circ {}^t \Gamma'_j. \end{aligned}$$

For each i , assume that $k_i^* D_i = \sum_l n_{i,l} Z'_{i,l}$, where $Z'_{i,l} \subset D_i$ are irreducible divisors. Let $Z_{i,l}$ be a resolution of $Z'_{i,l}$ and let $\Gamma_{i,l} \in \text{CH}_{d-1}(Z_{i,l} \times X)$ be the restriction of Γ'_i to $Z_{i,l}$. then we have

$$\Gamma'_i \circ k_i^* \circ k_{i,*} \circ {}^t \Gamma'_i = \sum_l n_{i,l} \Gamma_{i,l} \circ {}^t \Gamma_{i,l}.$$

For $i \neq j$, we set $Z_{\{i,j\}}$ to be the intersection of D_i and D_j . Let $\Gamma'_{i,j} \in \text{CH}_{d-1}(Z_{\{i,j\}} \times X)$ be the restriction of Γ'_i to $Z_{\{i,j\}}$. Thus we have

$$\Gamma'_i \circ k_i^* \circ k_{j,*} \circ {}^t \Gamma'_j = \Gamma'_{i,j} \circ {}^t \Gamma'_{j,i}.$$

It follows that

$$\Gamma'_i \circ k_i^* \circ k_{j,*} \circ {}^t \Gamma'_j + \Gamma'_j \circ k_j^* \circ k_{i,*} \circ {}^t \Gamma'_i = (\Gamma'_{i,j} + \Gamma'_{j,i}) \circ ({}^t \Gamma'_{i,j} + {}^t \Gamma'_{j,i}) - \Gamma'_{i,j} \circ {}^t \Gamma'_{i,j} - \Gamma'_{j,i} \circ {}^t \Gamma'_{j,i}$$

Hence $\Delta_X - X \times x - x \times X$ is of the given form. \square

The above result of Voisin is sufficient for our main application to cubic threefolds and cubic fourfolds. However, there is a more general version which we state below for future reference.

Proposition 3.9. *Let X be a smooth projective variety of dimension d over \mathbb{C} . Assume that $\text{CH}_i(X)$ is universally trivial for $i = 0, 1, \dots, r-1$. Then we have the following higher decomposition of diagonal*

$$\Delta_X = (X \times x + x \times X) + \Sigma + \Gamma, \quad \in \text{CH}_d(X \times X),$$

which satisfies the following conditions.

- (1) *The cycle $\Sigma \in \text{CH}^*(X) \otimes \text{CH}^*(X)$ is a completely decomposable symmetric d -cycle.*
- (2) *There are smooth projective varieties Z_i of dimension $d-2r$, correspondence $\Gamma_i \in \text{CH}_{d-r}(Z_i \times X)$ and integers n_i such that*

$$\Gamma = \sum_i n_i \Gamma_i \circ \sigma_i \circ {}^t \Gamma_i, \quad \text{in } \text{CH}_d(X \times X),$$

where $\sigma_i : Z_i \rightarrow Z_i$ is either the identity map or an involution. If $r = 1$, then all σ_i can be taken to be identity.

Proof. Since $\text{CH}_0(X)$ is universally trivial, we have a decomposition of the diagonal

$$\Delta_X = X \times x + \Gamma', \quad \text{in } \text{CH}_d(X \times X),$$

where Γ' is supported on $D \times X$ for some divisor D . We assume that $D = \cup_j D_j$. Then the restriction of Γ' to the generic point of D_j is a 1-cycle on $X_{\mathbb{C}(D_j)}$. By the universal triviality of $\text{CH}_1(X)$, there exist a 1-cycle $\gamma_j \in \text{CH}_1(X)$ such that $D_j \otimes \gamma_j$ agrees with Γ' over the generic point of D_j . Thus we have

$$\Delta_X = X \times x + \Gamma'_1 + \Gamma'', \quad \text{in } \text{CH}_d(X \times X),$$

where $\Gamma'_1 = \sum_j D_j \otimes \gamma_j$ and Γ'' is supported on $Y \times X$, where $Y \subset X$ is a closed subset of codimension 2. By repeating the above argument, we eventually get

$$\Delta_X = X \times x + \Gamma'_1 + \dots + \Gamma'_{r-1} + \Gamma''', \quad \text{in } \text{CH}_d(X \times X),$$

where $\Gamma'_i \in \text{CH}^i(X) \otimes \text{CH}_i(X)$, $1 \leq i \leq r-1$ and Γ''' is supported on $Z \times X$ for some closed subset $Z \subset X$ of codimension r . Then we carry out the symmetrization argument $\Delta_X = \Delta_X \circ {}^t \Delta_X$. We only need to note that $\Gamma'_i \circ (-)$ (resp. $(-) \circ {}^t \Gamma'_i$) is again of the form Γ'_i (resp. ${}^t \Gamma'_i$). Then we carry out a similar argument as before to show that $\Gamma''' \circ {}^t \Gamma'''$ is of the required form. First, there exist smooth projective varieties D_i of dimension $d-r$, correspondences $\Gamma_i''' \in \text{CH}_d(D_i \times X)$ and morphism $f_i : D_i \rightarrow X$ such that

$$\Gamma''' = \sum_i (f_i, \text{Id}_X)_* \Gamma_i''' = \sum_i \Gamma_i''' \circ f_i^*.$$

Then we have

$$\Gamma''' \circ {}^t \Gamma''' = \sum_{i,j} \Gamma_i''' \circ f_i^* \circ f_{j,*} \circ {}^t \Gamma_j'''$$

We write down the cycles $f_i^* \circ f_{j,*} \in \text{CH}_{d-2r}(D_i \times D_j)$ explicitly. Then the terms of the above sum can be grouped into the following types of terms.

Term type 1: $\Gamma_1 \circ {}^t\Gamma_2 + \Gamma_2 \circ {}^t\Gamma_1$, where $\Gamma_1, \Gamma_2 \in \text{CH}_{d-r}(Z \times X)$ for some smooth projective variety Z of dimension $d - 2r$. This form can be written as

$$(\Gamma_1 + \Gamma_2) \circ {}^t(\Gamma_1 + \Gamma_2) - \Gamma_1 \circ {}^t\Gamma_1 - \Gamma_2 \circ {}^t\Gamma_2.$$

Thus such a term can be written as the required form.

Term type 2: $\Gamma \circ {}^t\Gamma$, where $\Gamma \in \text{CH}_{d-r}(Z \times X)$ for some smooth projective variety Z of dimension $d - 2r$. Such a term is of the required form.

Term type 3: $\Gamma \circ \sigma \circ {}^t\Gamma$, where $\Gamma \in \text{CH}_{d-r}(Z \times X)$ for some smooth projective variety Z of dimension $d - 2r$ and $\sigma : Z \rightarrow Z$ is an involution. \square

Remark 3.10. In Proposition 3.8, only diagonalized terms $\Gamma_i \circ {}^t\Gamma_i$ appear since we are allowed to blow up X to make the D_i 's to be normal crossing. However, blow-up only preserves universal triviality of CH_0 and hence not allowed in Proposition 3.9. Thus the image of D_i in X can fail to be normal and that produces the terms $\Gamma_i \circ \sigma_i \circ {}^t\Gamma_i$ that are not diagonalized.

Corollary 3.11. *Let X be a smooth projective variety of dimension d over \mathbb{C} such that $\text{CH}_i(X)$ is universally trivial for all $i = 0, 1, \dots, r - 1$. Then the following are true.*

- (i) $\text{v.dim}(\text{CH}_r(X)) \leq d - 2r$.
- (ii) *There exist smooth projective varieties Z_i of dimension $d - 2r$, cycles $\Gamma_i \in \text{CH}_{d-r}(Z_i \times X)$ and integers $n_i \in \mathbb{Z}$ such that $\Gamma = \sum n_i \Gamma_i$ induces a universally surjective homomorphism $\bigoplus_i \text{CH}_0(Z_i) \rightarrow \text{CH}_r(X)$ and*

$$\sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{Z_i} = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in H^d(X, \mathbb{Z})_{\text{tr}}$, where $\sigma_i : Z_i \rightarrow Z_i$ is either the identity map or an involution.

- (iii) $H^{p,q}(X) = 0$ for all $p \neq q$ and $\min\{p, q\} < r$.

- (iv) $F^i H^d(X, \mathbb{Z}) = 0$, for all $i = 0, 1, \dots, 2r$.

Proof. Let $\gamma \in \text{CH}^r(X_L)$, then

$$\gamma = (\Delta_X)_* \gamma = \Sigma_* \gamma + \Gamma_* \gamma = \Gamma_* \gamma.$$

Since $\Gamma_* \gamma$ factors through $\sum \text{CH}_0((Z_i)_L)$, we have $\text{v.dim}(\gamma) \leq d - 2r$. Statement (ii) follows directly from the Proposition. Let $\omega \in H^{p,q}(X)$. Then we have

$$\omega = (\Delta_X)_* \omega = (x \times X + X \times x + \Sigma)_* \omega + \Gamma_* \omega,$$

where the first term vanishes whenever $p \neq q$. If $\max\{p, q\} < r$, then the second term also vanishes since it factors through $H^{p-r, q-r}(Z_i)$. Statement (iii) follows. Let $\alpha \in F^i H^d(X, \mathbb{Z})$ where $i \leq 2r$. Since Δ_X factors through varieties of dimension at most $d - 2r$, it follows that $\alpha = (\Delta_X)^* \alpha = 0$. Thus (iv) is proved. \square

Remark 3.12. The statement (iii) holds under the weaker assumption that $\text{CH}_i(X)_{\mathbb{Q}}$ is universally trivial for $i = 0, 1, \dots, r - 1$.

Theorem 3.13. *Let X be a smooth projective variety of dimension d over \mathbb{C} such that $\text{CH}_i(X)$ is universally trivial for all $i = 0, 1, \dots, r - 1$. Let F be a smooth projective variety and let $\gamma \in \text{CH}_r(X_{\mathbb{C}(F)})$ be a universally generating r -cycle. Then there exists a symmetric algebraic cycle $\theta \in \text{CH}_{d-2r}(F \times F)$ such that*

$$[\theta] \cdot (\hat{\alpha} \otimes \hat{\beta}) = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in F^{2r+1} H^d(X, \mathbb{Z})$, where $\hat{\alpha} := \Gamma^ \alpha$ for some spreading $\Gamma \in \text{CH}^r(F \times X)$ of γ .*

Proof. By Corollary 3.11, there exist smooth projective varieties Z_i of dimension $d - 2r$, cycles $\Gamma_i \in \text{CH}_{d-r}(Z_i \times X)$ and integers n_i such that

$$\sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{Z_i} = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in F^{2r+1} H^d(X, \mathbb{Z})$. Let $\gamma_i := \Gamma_i|_{\eta_{Z_i}} \in \text{CH}_r(X_{K_i})$, where $K_i = \mathbb{C}(Z_i)$. For each i , there exists $\tau_i \in \text{CH}_0(F_{K_i})$ such that $\Gamma_* \tau_i = \gamma_i$. Let $T_i \in \text{CH}_{d-2r}(Z_i \times F)$ be a spreading of τ_i and set

$$\theta = \sum n_i T_i \circ \sigma_i \circ {}^t T_i \in \text{CH}_{d-2r}(F \times F).$$

Thus we have

$$\begin{aligned}
[\theta] \cdot (\hat{\alpha} \otimes \hat{\beta}) &= \sum n_i [T_i \circ \sigma_i \circ {}^t T_i] \cup (\Gamma^* \alpha \otimes \Gamma^* \beta) \\
&= \sum n_i \langle T_i^* \Gamma^* \alpha, \sigma_i^* T_i^* \Gamma^* \beta \rangle_{Z_i} \\
&= \sum n_i \langle \Gamma_i'^* \alpha, \sigma_i^* \Gamma_i'^* \beta \rangle_{Z_i} \\
&= \sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{Z_i} \\
&= \langle \alpha, \beta \rangle_X,
\end{aligned}$$

where $\Gamma_i' = \Gamma \circ T_i \in \text{CH}_{d-r}(Z_i \times X)$ is a spreading of γ_i . Note that $\Gamma_i - \Gamma_i'$ vanishes over the generic point of Z_i and hence factors through a divisor of Z_i . It follows that $\Gamma_i^* \alpha = \Gamma_i'^* \alpha$ for all $\alpha \in F^{2r+1} \text{H}^d(X, \mathbb{Z})$, which is used to get the fourth equality above. \square

4. UNIVERSAL GENERATION OF 1-CYCLES ON CUBIC HYPERSURFACES

In this section we show that the Chow group of 1-cycles on a smooth cubic hypersurface is universally generated by lines. Let $X \subseteq \mathbb{P}_K^{d+1}$ be a smooth cubic hypersurface of dimension d over an arbitrary field K . Let $F = F(X)$ be the variety of lines on X and let

$$\begin{array}{ccc}
P & \xrightarrow{q} & X \\
p \downarrow & & \\
F & &
\end{array}$$

be the universal line.

Theorem 4.1. *Assume that $\text{CH}_0(F)$ contains an element of degree 1. Then*

$$P_* = q_* p^* : \text{CH}_0(F) \longrightarrow \text{CH}_1(X)$$

is universally surjective. Namely $P|_{\eta_F} \in \text{CH}_1(X_{K(F)})$ is universally generating.

The key ingredient one need to prove the above universal generation is the following relations among 1-cycles on X .

Proposition 4.2 ([18, 19]). *Let $\gamma_1, \gamma_2 \in \text{CH}_1(X)$ be 1-cycles of degree e_1 and e_2 respectively. Let $h \in \text{CH}^1(X)$ be the class of a hyperplane section.*

(i) *There exists a 0-cycle $\gamma \in \text{CH}_0(F)$ such that*

$$(2e_1 - 3)\gamma_1 + q_* p^* \gamma = ah^{d-1}, \quad \text{in } \text{CH}_1(X),$$

for some integer a . If γ_1 is represented by a geometrically irreducible curve C in general position, then γ can be take to be all the lines that meet C in two points.

(ii) *We have*

$$2e_2\gamma_1 + 2e_1\gamma_2 + q_* p^* \gamma' = 3e_1e_2 h^{d-1}, \quad \text{in } \text{CH}_1(X),$$

where $\gamma' = p_ q^* \gamma_1 \cdot p_* q^* \gamma_2 \in \text{CH}_0(F)$ is a 0-cycle of degree $5e_1e_2$.*

(iii) *Let $\xi \in \text{CH}^r(X)$ with $r < d - 1$. Then*

$$2e_1\xi + q_* p^* \gamma'' = bh^r, \quad \text{in } \text{CH}^r(X),$$

where $\gamma'' = p_ q^* \gamma_1 \cdot p_* q^* \xi \in \text{CH}^{d+r-3}(F)$ and $b \in \mathbb{Z}$.*

We now prove Theorem 4.1 assuming Proposition 4.2.

Proof of Theorem 4.1. Since our base field K is arbitrary, we only need to show that $q_* p^* : \text{CH}_0(F) \rightarrow \text{CH}_1(X)$ is surjective. Let $\mathbf{a}_0 \in \text{CH}_0(F)$ be an element of degree 1. Take $\mathbf{l} = q_* p^* \mathbf{a}_0 \in \text{CH}_1(X)$. Let $\gamma \in \text{CH}_1(X)$ be a 1-cycle of degree e . Then by Proposition 4.2 (i), we see that $(2e - 3)\gamma$ expressed as the sum of an element of $q_* p^* \text{CH}_0(F)$ and a multiple of h^{d-1} . Similary Proposition 4.2 (ii) applied to γ and \mathbf{l} , we see that 2γ is contained in $q_* p^* \text{CH}_0(F) + \mathbb{Z}h^{d-1}$. Thus $\gamma \in q_* p^* \text{CH}_0(F) + \mathbb{Z}h^{d-1}$. We only need to show that h^{d-1} is contained in $q_* p^* \text{CH}_0(F)$. Apply Proposition 4.2 (ii) to $\gamma_1 = \gamma_2 = h^{d-1}$, we get

$$6h^{d-1} + 6h^{d-1} + q_* p^* \mathbf{a}_1 = 27h^{d-1}, \quad \mathbf{a}_1 \in \text{CH}_0(F).$$

It follows that $15h^{d-1} = q_*p^*\mathbf{a}_1$. We apply the proposition again to h^{d-1} and \mathbf{l} and get

$$2h^{d-1} + 6\mathbf{l} + q_*p^*\mathbf{a}_2 = 9h^{d-1}, \quad \mathbf{a}_2 \in \mathrm{CH}_0(F).$$

Thus $7h^{d-1} = q_*p^*(\mathbf{a}_2 - 6\mathbf{a}_0)$. Thus we conclude that $h^{d-1} \in q_*p^*\mathrm{CH}_0(F)$. \square

The proof of Proposition 4.2 over an algebraically closed field is given in [18]. In this section, we explore the techniques in [18] in a more general setting and give a detailed proof of Proposition 4.2 over arbitrary base. Let B be a base scheme and let $\mathcal{X} \rightarrow B$ be a smooth (over B) cubic hypersurface of dimension d . This means that there exists an embedding $\mathcal{X} \hookrightarrow \mathbb{P}_B^{d+1}$ defined over B such that all fibers are smooth cubic hypersurfaces in \mathbb{P}^{d+1} . We would like to study the geometry of $\mathcal{X}^{[2]}$, the relative Hilbert scheme of two points on \mathcal{X} . Let $\delta \in \mathrm{CH}^1(\mathcal{X}^{[2]})$ be the “half boundary”. Namely, 2δ is the class of the boundary divisor parametrizing nonreduced length-2 subschemes. Let \mathcal{F}/B be the relative variety of lines on \mathcal{X} and let

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{q} & \mathcal{X} \\ p \downarrow & & \\ \mathcal{F} & & \end{array}$$

be the universal family of lines. It is known by [2] that \mathcal{F}/B is smooth of relative dimension $2d - 4$.

Following Galkin–Shinder [8], we define a rational map

$$\Phi : \mathcal{X}^{[2]} \dashrightarrow P_{\mathcal{X}} := \mathbb{P}(\mathcal{T}_{\mathbb{P}_B^{d+1}/B}|_{\mathcal{X}})$$

as follows. Let $x, y \in \mathcal{X}$ be two points over $b \in B$ and they determine a line $L_{x,y} \subset \mathbb{P}_{\kappa(b)}^{d+1}$. If $L_{x,y}$ is not contained in \mathcal{X}_b , then $L_{x,y}$ intersects \mathcal{X}_b in a third point $z \in \mathcal{X}_b$. Then $\varphi([x, y])$ is the point represented by the 1-dimensional subspace $\mathcal{T}_{L_{x,y},z}$ in $\mathbb{P}(\mathcal{T}_{\mathbb{P}_b^{d+1},z})$. Let $\mathcal{P}_{\mathcal{F}}^{[2]} \subset \mathcal{X}^{[2]}$ be the relative Hilbert scheme of \mathcal{P}/\mathcal{F} . Let $p' : \mathcal{P}_{\mathcal{F}}^{[2]} \rightarrow \mathcal{F}$ be the structure morphism. Voisin [29] showed that the indeterminacy of Φ can be resolved by blowing up $\mathcal{X}^{[2]}$ along $\mathcal{P}_{\mathcal{F}}^{[2]}$. The resulting morphism $\tilde{\Phi} : \widetilde{\mathcal{X}^{[2]}} \rightarrow P_{\mathcal{X}}$ is the blow up of $P_{\mathcal{X}}$ along $\mathcal{P} \subset P_{\mathcal{X}}$. The inclusion $i_1 : \mathcal{P} \hookrightarrow P_{\mathcal{X}}$ sends a point $(x \in l)$ to the direction $\mathcal{T}_{l,x}$ in $\mathcal{T}_{\mathcal{X}/B,x}$. To summarize, we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\pi_1} & \mathcal{P} & & \\ & \searrow j & \downarrow i_1 & & \\ & & \widetilde{\mathcal{X}^{[2]}} & \xrightarrow{\tilde{\Phi}} & P_{\mathcal{X}} \\ \pi_2 \downarrow & & \downarrow \tau & \nearrow \Phi & \\ \mathcal{P}_{\mathcal{F}}^{[2]} & \xrightarrow{i_2} & \mathcal{X}^{[2]} & & \end{array}$$

We also have the natural identification

$$\mathcal{E} = \mathcal{P} \times_{\mathcal{F}} \mathcal{P}_{\mathcal{F}}^{[2]}$$

and the morphisms π_1 and π_2 are the two projections.

There is a double cover $\sigma : \widetilde{\mathcal{X} \times_B \mathcal{X}} \rightarrow \mathcal{X}^{[2]}$, where $\widetilde{\mathcal{X} \times_B \mathcal{X}}$ is the blow up of $\mathcal{X} \times_B \mathcal{X}$ along the diagonal. There is also a morphism

$$\Psi : \widetilde{\mathcal{X} \times_B \mathcal{X}} \rightarrow P_{\mathcal{X}}, \quad (x, y) \mapsto [\mathcal{T}_{L_{x,y},x}].$$

Note that the composition $\widetilde{\mathcal{X} \times_B \mathcal{X}} \rightarrow P_{\mathcal{X}} \rightarrow \mathcal{X}$, is the blow-up morphism followed by the projection onto the first factor. The above morphisms form the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{X}^{[2]} & \xleftarrow{\sigma} & \widetilde{\mathcal{X} \times_B \mathcal{X}} & \xrightarrow{\Psi} & P_{\mathcal{X}} \\ & & \downarrow \rho & & \downarrow \pi \\ & & \mathcal{X} \times_B \mathcal{X} & \xrightarrow{p_1} & \mathcal{X} \end{array}$$

Given algebraic cycles $\alpha, \beta \in \text{CH}_*(\mathcal{X})$, we write

$$\alpha \hat{\otimes} \beta := \sigma_* \rho^* (\alpha \times_B \beta) \in \text{CH}_*(\mathcal{X}^{[2]}).$$

Two points $x, y \in \mathcal{X}_b$ determine a line $L_{x,y}$ in \mathbb{P}_b^{d+1} . This defines a morphism

$$\varphi : \mathcal{X}^{[2]} \longrightarrow \mathcal{G}(2, \mathcal{O}_B^{d+2}),$$

where $\mathcal{G}(2, \mathcal{O}_B^{d+2})$ is the relative Grassmannian of rank two subbundles of \mathcal{O}_B^{d+2} . Together with the previous morphisms, we have a commutative diagram with all squares being fiber products.

$$(7) \quad \begin{array}{ccccc} \widetilde{\mathcal{X} \times_B \mathcal{X} \cup \mathcal{X}^{[2]}} & \xrightarrow{\quad} & P_{\mathcal{X}} & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow & & \downarrow i' & & \downarrow i \\ \mathcal{Q} & \xrightarrow{\varphi'} & \mathcal{G}(1, 2, \mathcal{O}_B^{d+2}) & \xrightarrow{\tilde{\pi}} & \mathbb{P}_B^{d+1} \\ \downarrow f & & \downarrow \bar{f} & & \\ \mathcal{X}^{[2]} & \xrightarrow{\varphi} & \mathcal{G}(2, \mathcal{O}_B^{d+2}) & & \end{array}$$

Let $h \in \text{CH}^1(\mathbb{P}_B^{d+1})$ be the class of a hyperplane section. We still use $h \in \text{CH}^1(\mathcal{X})$ to denote the restriction of h to \mathcal{X} and let $h_{\mathcal{Q}} \in \text{CH}^1(\mathcal{Q})$ be the pull-back of h to \mathcal{Q} via the natural morphism $\mathcal{Q} \rightarrow \mathbb{P}_B^{d+1}$. Let $\mathcal{E} \subset \mathcal{O}_B^{d+2}$ be the tautological rank-2 subbundle on $\mathcal{G}(2, \mathcal{O}_B^{d+2})$.

Lemma 4.3. *We have*

$$\mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{\mathcal{X}^{[2]}}} - \tau^*(2h \hat{\otimes} 1 - 3\delta)$$

in $\text{CH}^1(\widetilde{\mathcal{X}^{[2]}})$.

Proof. As divisors on \mathcal{Q} , we have

$$\widetilde{\mathcal{X}^{[2]}} = h_{\mathcal{Q}} + f^* \mathbf{a}, \quad \text{in } \text{CH}^1(\mathcal{Q}),$$

for some $\mathbf{a} \in \text{CH}^1(\mathcal{X}^{[2]})$. Note that \mathcal{Q} is a \mathbb{P}^1 -bundle over $\mathcal{X}^{[2]}$ and we have the following short exact sequence

$$0 \longrightarrow \mathcal{O}(h_{\mathcal{Q}}) \otimes \frac{f^*(\mathcal{E}|_{\mathcal{X}^{[2]}})}{\mathcal{O}(-h_{\mathcal{Q}})} \longrightarrow \mathcal{T}_{\mathcal{Q}/B} \longrightarrow f^* \mathcal{T}_{\mathcal{X}^{[2]}/B} \longrightarrow 0.$$

As a consequence, we have

$$\begin{aligned} K_{\mathcal{Q}/B} &= -c_1(\mathcal{T}_{\mathcal{Q}/B}) \\ &= f^* K_{\mathcal{X}^{[2]}/B} - 2h_{\mathcal{Q}} - f^* c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}). \end{aligned}$$

The relative canonical divisor of $\widetilde{\mathcal{X}^{[2]}}$ can be computed by the adjunction formula as

$$\begin{aligned} K_{\widetilde{\mathcal{X}^{[2]}/B} &= (K_{\mathcal{Q}/B} + \widetilde{\mathcal{X}^{[2]}})|_{\widetilde{\mathcal{X}^{[2]}}} \\ &= \tau^* K_{\mathcal{X}^{[2]}/B} - h_{\mathcal{Q}}|_{\widetilde{\mathcal{X}^{[2]}}} - \tau^* c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}) + \tau^* \mathbf{a}. \end{aligned}$$

Since $\widetilde{\mathcal{X}^{[2]}}$ is the blow-up of $\mathcal{X}^{[2]}$, we also have

$$K_{\widetilde{\mathcal{X}^{[2]}/B} = \tau^* K_{\mathcal{X}^{[2]}/B} + \mathcal{E}.$$

Comparing the above two expressions, we get

$$(8) \quad \mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{\mathcal{X}^{[2]}}} - \tau^* c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}) + \tau^* \mathbf{a}, \quad \text{in } \text{CH}^1(\widetilde{\mathcal{X}^{[2]}}).$$

To determine the value of \mathbf{a} , we apply τ_* to the above equation and get

$$(9) \quad 0 = -f_*(h_{\mathcal{Q}} \cdot \widetilde{\mathcal{X}^{[2]}}) - c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}) + \mathbf{a}$$

in $\mathrm{CH}^1(\mathcal{X}^{[2]})$. The first term on the right hand side can be computed as follows.

$$\begin{aligned} f_*(h_{\mathcal{Q}} \cdot \widetilde{\mathcal{X}^{[2]}}) &= f_* \left(h_{\mathcal{Q}} \cdot (\widetilde{\mathcal{X}^{[2]}} + \widetilde{\mathcal{X} \times_B \mathcal{X}}) \right) - f_* \left(h_{\mathcal{Q}} \cdot \widetilde{\mathcal{X} \times_B \mathcal{X}} \right) \\ &= f_*(h_{\mathcal{Q}} \cdot 3h_{\mathcal{Q}}) - \sigma_* \rho^*(h \otimes 1) \\ &= 3f_* h_{\mathcal{Q}}^2 - h \hat{\otimes} 1. \end{aligned}$$

Note that $\mathcal{Q} = \mathbb{P}(\mathcal{E}|_{\mathcal{X}^{[2]}})$ is a \mathbb{P}^1 -bundle over $\mathcal{X}^{[2]}$ and $h_{\mathcal{Q}}$ is the class of the associated relative $\mathcal{O}(1)$ -bundle. Thus we have the equation

$$h_{\mathcal{Q}}^2 + f^* c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}) \cdot h_{\mathcal{Q}} + f^* c_2(\mathcal{E}|_{\mathcal{X}^{[2]}}) = 0$$

and it follows that

$$f_* h_{\mathcal{Q}}^2 = -c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}).$$

This combined with equation (9), we get

$$\mathbf{a} = -2c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}) - h \hat{\otimes} 1.$$

We plug this into (8) and get

$$(10) \quad \mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{\mathcal{X}^{[2]}}} - \tau^*(h \hat{\otimes} 1) - 3\tau^* c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}), \quad \text{in } \mathrm{CH}^1(\widetilde{\mathcal{X}^{[2]}}).$$

We still need to compute $c_1(\mathcal{E}|_{\mathcal{X}^{[2]}})$. For simplicity, we write $\mathcal{V} = \mathcal{O}_B^{d+2}$. For any coherent sheaf \mathcal{F} on \mathcal{X} , we define

$$\mathcal{F}^{[2]} := \sigma_* \rho^* p_1^* \mathcal{F}.$$

When \mathcal{F} is locally free of rank r , we have $\mathcal{F}^{[2]}$ being locally free of rank $2r$. The inclusion $\mathcal{O}_{\mathcal{X}}(-1) \hookrightarrow \mathcal{V}$ induces an inclusion

$$(11) \quad \mathcal{O}(-1)^{[2]} \hookrightarrow \mathcal{V} \otimes \mathcal{O}_{\mathcal{X}}^{[2]} = \mathcal{V} \otimes \sigma_* \mathcal{O}_{\widetilde{\mathcal{X} \times_B \mathcal{X}}}.$$

Recall that $\delta \in \mathrm{CH}^1(\mathcal{X}^{[2]})$ is the “half boundary” which fits into the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}^{[2]}} \longrightarrow \sigma_* \mathcal{O}_{\widetilde{\mathcal{X} \times_B \mathcal{X}}} \longrightarrow \mathcal{O}_{\mathcal{X}^{[2]}}(-\delta) \longrightarrow 0.$$

Combining with the inclusion (11), we get

$$\mathcal{O}(-1)^{[2]} \otimes \mathcal{O}(\delta) \rightarrow \mathcal{V},$$

which defines the morphism φ . Namely,

$$\varphi^* \mathcal{E} \cong \mathcal{O}(-1)^{[2]} \otimes \mathcal{O}(\delta).$$

The Grothendieck–Riemann–Roch formula gives

$$c_1(\mathcal{O}(-1)^{[2]}) = -h \hat{\otimes} 1 - \delta.$$

It follows that

$$c_1(\mathcal{E}|_{\mathcal{X}^{[2]}}) = c_1(\varphi^* \mathcal{E}) = -h \hat{\otimes} 1 + \delta$$

We combine this with equation (10) and prove the lemma. \square

Lemma 4.4. *For any algebraic cycle $\Gamma \in \mathrm{CH}^r(\mathcal{X}^{[2]})$, the following are true.*

(i) *The following equation holds in $\mathrm{CH}^{r+1}(P_{\mathcal{X}})$,*

$$\tilde{\Phi}_*(\mathcal{E} \cdot \tau^* \Gamma) = (i_1)_* p^* \gamma,$$

where $\gamma = p'_(i_2)^* \Gamma$ in $\mathrm{CH}^{r-2}(\mathcal{F})$. In particular, we have*

$$\pi_* \tilde{\Phi}_*(\mathcal{E} \cdot \tau^* \Gamma) = q_* p^* \gamma, \quad \text{in } \mathrm{CH}^{r-3}(\mathcal{X}).$$

(ii) *We have*

$$\tilde{\Phi}_* \tau^* \Gamma = (i')^* \tilde{f}^* \varphi_* \Gamma - \Psi_* \sigma^* \Gamma, \quad \text{in } \mathrm{CH}^r(P_{\mathcal{X}}).$$

Proof. The statement (i) follows from a straightforward calculation as follows.

$$\begin{aligned}\tilde{\Phi}_*(\mathcal{E} \cdot \tau^* \Gamma) &= \tilde{\Phi}_* j_* j^* \tau^* \Gamma \\ &= (i_1)_* (\pi_1)_* \pi_2^* i_2^* \Gamma \\ &= (i_1)_* p^* p'_* i_2^* \Gamma.\end{aligned}$$

We also have

$$\begin{aligned}\tilde{\Phi}_* \tau^* \Gamma &= \left(\tilde{\Phi}_* \tau^* \Gamma + \Psi_* \sigma^* \Gamma \right) - \Psi_* \sigma^* \Gamma \\ &= (i')^* \tilde{f}^* \varphi_* \Gamma - \Psi_* \sigma^* \Gamma.\end{aligned}$$

This proves (ii). \square

Lemma 4.5. *Let $\{\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_n, \tilde{\Sigma}_1, \dots, \tilde{\Sigma}_m\}$ be a set of distinct irreducible closed subschemes of codimension $d-1$ in \mathcal{X} which are all reduced and dominate B . Let $\tilde{\Gamma} = \sum n_k \tilde{\Gamma}_k$ and $\tilde{\Sigma} = \sum m_k \tilde{\Sigma}_k$ be two algebraic cycles of codimension $d-1$ in \mathcal{X} . Let e (resp. e') be the degree of $\tilde{\Gamma}|_{\mathcal{X}_{\eta_B}}$ (resp. $\tilde{\Sigma}|_{\mathcal{X}_{\eta_B}}$) as a 1-cycle on \mathcal{X}_{η_B} .*

(i) *There exists an algebraic cycle Γ on $\mathcal{X}^{[2]}$ such that*

$$\begin{aligned}\pi_* \Psi_* \sigma^* \Gamma &= 0, \\ \pi_* \Psi_* \sigma^* (h \hat{\otimes} 1 \cdot \Gamma) &= e \tilde{\Gamma}, \\ \pi_* \Psi_* \sigma^* (\delta \cdot \Gamma) &= \sum n_k^2 \tilde{\Gamma}_k.\end{aligned}$$

(ii) *There exists an algebraic cycle Γ' on $\mathcal{X}^{[2]}$ such that*

$$\begin{aligned}\pi_* \Psi_* \sigma^* \Gamma' &= 0, \\ \pi_* \Psi_* \sigma^* (h \hat{\otimes} 1 \cdot \Gamma') &= e' \tilde{\Gamma} + e \tilde{\Sigma}, \\ \pi_* \Psi_* \sigma^* (\delta \cdot \Gamma') &= 0.\end{aligned}$$

(iii) *Let $\tilde{\Xi}$ be an algebraic cycle of codimension $r < d-1$ on \mathcal{X} . Assume that all components of $\tilde{\Xi}$ dominate B . Then there exists an algebraic cycle Γ'' on $\mathcal{X}^{[2]}$ such that*

$$\begin{aligned}\pi_* \Psi_* \sigma^* \Gamma'' &= 0 \\ \pi_* \Psi_* \sigma^* (h \hat{\otimes} 1 \cdot \Gamma'') &= e \tilde{\Xi}, \\ \pi_* \Psi_* \sigma^* (\delta \cdot \Gamma'') &= 0.\end{aligned}$$

Proof. Let $\tilde{\Gamma}_{kl} \subset \mathcal{X} \times_B \mathcal{X}$ be the component of $\tilde{\Gamma}_k \times_B \tilde{\Gamma}_l$ that dominates B . Let $\Gamma_{kl} \subset \widetilde{\mathcal{X} \times_B \mathcal{X}}$ be the strict transform of $\tilde{\Gamma}_{kl}$. Then there exists an algebraic cycle Γ on $\mathcal{X}^{[2]}$ such that $\sigma^* \Gamma = \sum n_k n_l \Gamma_{kl}$. Thus

$$\pi_* \Psi_* \sigma^* \Gamma = \sum_{k,l} n_k n_l (p_1)_* \rho_* \Gamma_{kl} = 0$$

and

$$\pi_* \Psi_* \sigma^* (\Gamma \cdot h \hat{\otimes} 1) = \sum n_k n_l (p_1)_* \rho_* \left(\Gamma_{kl} \cdot \rho^* (h \otimes 1 + 1 \otimes h) \right) = \sum n_k n_l e_k \tilde{\Gamma}_l = e \tilde{\Gamma},$$

where $e_k = \deg(\tilde{\Gamma}_k|_{\mathcal{X}_{\eta_B}})$, $e = \sum n_k e_k$. The remaining equation holds because

$$\pi_* \Psi_* (\Gamma_{kl} \cdot \sigma^* \delta) = \begin{cases} 0, & \text{if } k \neq l; \\ \tilde{\Gamma}_k, & \text{if } k = l. \end{cases}$$

This proves (i). Statement (ii) and (iii) are proved similarly. \square

Proposition 4.6. *Let $\tilde{\Gamma}$ and $\tilde{\Sigma}$ be two distinct irreducible closed subschemes of codimension $d-1$ in \mathcal{X} which are reduced and both dominate B . Let e (resp. e') be the degree of $\tilde{\gamma} := \tilde{\Gamma}|_{\mathcal{X}_{\eta_B}}$ (resp. $\tilde{\sigma} := \tilde{\Sigma}|_{\mathcal{X}_{\eta_B}}$) as a 1-cycle on \mathcal{X}_{η_B} .*

(i) There exists $\gamma \in \mathrm{CH}^{2d-4}(\mathcal{F})$ such that

$$(2e-3)\tilde{\Gamma} + q_*p^*\gamma = h \cdot i^*\mathbf{a}, \quad \text{in } \mathrm{CH}^{d-1}(\mathcal{X})$$

for some $\mathbf{a} \in \mathrm{CH}^{d-2}(\mathbb{P}_B^{d+1})$.

(ii) There exists $\gamma' \in \mathrm{CH}^{2d-4}(\mathcal{F})$ such that

$$2e'\tilde{\Gamma} + 2e\tilde{\Sigma} + q_*p^*\gamma' = h \cdot i^*\mathbf{b}, \quad \text{in } \mathrm{CH}^{d-1}(\mathcal{X}),$$

for some $\mathbf{b} \in \mathrm{CH}^{d-2}(\mathbb{P}_B^{d+1})$. Moreover $\gamma'|_{\eta_B} = p_*q^*\tilde{\gamma} \cdot p_*q^*\tilde{\sigma}$ which is of degree $5e'e$ over η_B .

(iii) Let $\tilde{\Xi}$ be an algebraic cyce of codimension $r > d-1$, all of whose components dominate B . Then there exists $\gamma'' \in \mathrm{CH}^{d+r-3}(\mathcal{F})$ such that

$$2e\tilde{\Xi} + q_*p^*\gamma'' = h \cdot i^*\mathbf{c}, \quad \text{in } \mathrm{CH}^r(\mathcal{X}),$$

for some $\mathbf{c} \in \mathrm{CH}^{r-1}(\mathbb{P}_B^{d+1})$. Moreover $\gamma''|_{\eta_B} = p_*q^*\tilde{\xi} \cdot p_*q^*\tilde{\gamma}$, where $\tilde{\xi} = \tilde{\Xi}|_{\mathcal{X}_{\eta_B}}$.

Proof. By Lemma 4.5, we can find a cycle Γ on $\mathcal{X}^{[2]}$ with the properties specified in Lemma 4.5. We apply $\pi_*\tilde{\Phi}_*(-) \cdot \tau^*\Gamma$ to the equality in Lemma 4.3 and get

$$\begin{aligned} \pi_*\tilde{\Phi}_*(\mathcal{E} \cdot \tau^*\Gamma) &= -\pi_*\tilde{\Phi}_*(h_{\mathcal{Q}}|_{\widetilde{\mathcal{X}^{[2]}}} \cdot \tau^*\Gamma) - \pi_*\tilde{\Phi}_*\tau^*((2h\otimes 1 - 3\delta) \cdot \Gamma) \\ &= -h \cdot \pi_*\tilde{\Phi}_*\tau^*\Gamma - (2e-3)\tilde{\Gamma} \\ &= -h \cdot \pi_*(i')^*\tilde{f}^*\varphi_*\Gamma + h \cdot \Psi_*\sigma^*\Gamma - (2e-3)\tilde{\Gamma} \\ &= -h \cdot i^*\tilde{\pi}_*\tilde{f}^*\varphi_*\Gamma - (2e-3)\tilde{\Gamma}. \end{aligned}$$

Here we use Lemma 4.5 in the second equality and Lemma 4.4 (ii) in the third equality. Take $\mathbf{a} = -\tilde{\pi}_*\tilde{f}^*\varphi_*\Gamma$ and apply Lemma 4.4 (i) and the statement (i) follows. Statement (ii) is proved similarly except the degree of γ' . This degree can be computed after base change to the geometry generic point of B . We can also move the cycles to general position. Chasing through the proof of Lemma 4.4 (i), we see that γ' can be represented by the lines meets both $\tilde{\Gamma}$ and $\tilde{\Sigma}$. The degree is shown to be $5e'e$ in [18]. Statement (iii) is proved similarly. \square

Proof of Proposition 4.2. Now we take $B = \mathrm{Spec}(K)$. Proposition 4.6 (ii) implies Proposition 4.2 (ii) in the case of γ_1 and γ_2 being two distinct curves. Note that the equality of Proposition 4.2 (ii) is bilinear in both γ_1 and γ_2 and we can also apply the moving lemma to either γ_1 or γ_2 . This shows that statement (ii) is true in full generality. One proves (iii) similarly. We apply Proposition 4.6 and see that Proposition 4.2 (i) holds for γ_1 being an irreducible curve. Assume that (i) holds for γ_1 and γ'_1 . Then we have

$$(12) \quad (2e_1-3)\gamma_1 + \cdots = \cdots$$

$$(13) \quad (2e_2-3)\gamma'_1 + \cdots = \cdots$$

Here we omit the term coming from F on the left hand side and the term of a multiple of h^r on the right hand side; we do the same in the remaining part of this proof. Apply (ii) to the pair (γ_1, γ'_1) and get

$$2e'_1\gamma_1 + 2e_1\gamma'_1 + \cdots = \cdots$$

Combine the above equations and get

$$(2(e_1 + e'_1) - 3)(\gamma_1 + \gamma'_1) + \cdots = \cdots$$

Thus the quality holds for $\gamma_1 + \gamma'_1$ and it follows that (i) holds for all effective 1-cycles. We still need to show (i) for $-\gamma_1$. For that we apply (ii) to the pair (γ_1, γ_1) and get

$$4e_1\gamma_1 + \cdots = \cdots$$

Subtract this from equation (12), we have

$$(-2e_1-3)\gamma_1 + \cdots = \cdots$$

This establishes (i) for $-\gamma_1$ and hence for all 1-cycles. \square

5. CUBIC HYPERSURFACES OF SMALL DIMENSIONS

This section is devoted to applications of the universal generation result of the previous section. We relate the rationality problem of a cubic hypersurface of small dimension to the geometry of its variety of lines.

5.1. A special decomposition of the diagonal. In this subsection we fix a smooth cubic hypersurface $X \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of dimension $d = 3$ or 4 . Let $h \in \text{CH}^1(X)$ be the class of a hyperplane section.

Theorem 5.1. *Assume that X admits a Chow theoretical decomposition of the diagonal. Then the following holds.*

(i) *If $d = 3$, then there exists a symmetric 1-cycle $\theta \in \text{CH}_1(F \times F)$ such that*

$$\Delta_X = x \times X + X \times x + (\gamma \times h + h \times \gamma) + (P \times P)_* \theta, \quad \text{in } \text{CH}^3(X \times X),$$

where $\gamma \in \text{CH}_1(X)$.

(ii) *If $d = 4$, then there exists a symmetric 2-cycle $\theta \in \text{CH}_2(F \times F)$ such that*

$$\Delta_X = x \times X + X \times x + \Sigma + (P \times P)_* \theta, \quad \text{in } \text{CH}^4(X \times X),$$

where $\Sigma \in \text{CH}_2(X) \otimes \text{CH}_2(X)$ is a symmetric decomposable cycle. Moreover, Σ can be chosen to be zero if $\text{Hdg}^4(X) = \mathbb{Z}h^2$.

Proof. The construction of θ as in Theorem 3.13 suffices in the case of $d = 3$. However, it is insufficient in the case of $d = 4$ since it produces correspondences factoring through curves. Using an argument of Voisin [29], we can show that these problematic terms can be absorbed into the terms $(P \times P)_* \theta$ and Σ . We make it more precise.

Assume that $d = 3$. Then by Proposition 3.8, there exist curves Z_i , correspondences $\Gamma_i \in \text{CH}^2(Z_i \times X)$ and integers n_i such that

$$\Delta_X = x \times X + X \times x + \sum n_i \Gamma_i \circ {}^t \Gamma_i, \quad \text{in } \text{CH}^3(X \times X).$$

As in the proof of Theorem 3.13, we use the universal generation of $\text{CH}_1(X)$ by lines to get $T_i \in \text{CH}^2(Z_i \times F)$ such that $\Gamma'_i := P \circ T_i \in \text{CH}^2(Z_i \times X)$ agrees with Γ_i over the generic point of Z_i . It follows that $\Gamma_i \circ {}^t \Gamma_i - \Gamma'_i \circ {}^t \Gamma'_i$ is a decomposable cycle, *i.e.* of the form $\gamma_i \otimes h + h \otimes \gamma_i$ for some $\gamma_i \in \text{CH}_1(X)$. Thus we have

$$\begin{aligned} \Delta_X &= x \times X + X \times x + \sum n_i \Gamma_i \circ {}^t \Gamma_i \\ &= x \times X + X \times x + \sum n_i \Gamma'_i \circ {}^t \Gamma'_i + \sum n_i (\gamma_i \otimes h + h \otimes \gamma_i) \\ &= x \times X + X \times x + (P \times P)_* \left(\sum n_i T_i \circ {}^t T_i \right) + \gamma \otimes h + h \otimes \gamma, \quad \gamma = \sum n_i \gamma_i. \end{aligned}$$

Statement (i) follows by taking $\theta = \sum n_i T_i \circ {}^t T_i$.

Assume that $d = 4$, then we can obtain surfaces Z_i and correspondence $T_i \in \text{CH}_3(Z_i \times X)$ similarly. Then we conclude that the cohomology class of

$$\Delta_X - x \times X - X \times x - (P \times P)_* \theta$$

is decomposable. Since the integral Hodge conjecture holds for X (see Voisin [26]), we know that

$$\Gamma := \Delta_X - x \times X - X \times x - (P \times P)_* \theta - \gamma \otimes h - h \otimes \gamma - \Sigma = 0, \quad \text{in } \text{H}^4(X \times X, \mathbb{Z}),$$

for some $\gamma \in \text{CH}_1(X)$ and some symmetric cycle $\Sigma \in \text{CH}_2(X) \otimes \text{CH}_2(X)$. Since $\text{CH}_0(F) \rightarrow \text{CH}_1(X)$ is surjective and there exists $\tau \in \text{CH}_2(F)$ such that $P_* \tau = h$. Thus the term $\gamma \otimes h + h \otimes \gamma$ can be absorbed into the term $(P \times P)_* \theta$ and hence we can assume that $\gamma = 0$. There also exists $\tau' \in \text{CH}_1(F)$ such that $P_* \tau' = h$ in cohomology. Thus we can assume $\Sigma = 0$ if $\text{Hdg}^4(X) = \mathbb{Z}h^2$. By Proposition 5.3 (ii) below, we can modify θ by a homologically trivial cycle supported on the diagonal of $F \times F$ and assume that Γ is algebraically trivial. Now by [24, 25], we know that $\Gamma^{\circ N} = 0$ in $\text{CH}^4(X \times X)$ for some sufficiently large integer N . In the expansion of this equation, any term involving Σ is again decomposable of the same form and any power of $(P \times P)_* \theta$ is again of this form. After modifying θ and Σ , we get the equation

$$\Delta_X = x \times X + X \times x + \Sigma + (P \times P)_* \theta, \quad \text{in } \text{CH}^4(X \times X).$$

This finishes the proof. \square

Remark 5.2. When $d = 3$, the term $\gamma \otimes h + h \otimes \gamma$ can not be absorbed into θ . This is because the homomorphism $q_* p^* : H^2(F, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ is not surjective. Given any 0-cycle $\mathbf{o}_F \in \text{CH}_0(F)$ of degree 1, we set $\mathbf{l} = q_* p^* \mathbf{o}_F \in \text{CH}_1(X)$. Then the cycle γ can always be chosen to be a multiple of \mathbf{l} . This can be seen as follows. For any $\gamma' \in \text{CH}_1(X)$ with $\deg(\gamma') = 0$, then there exists $\gamma'' \in \text{CH}_1(X)$ such that $\gamma' = 5\gamma''$. Thus $\gamma' \otimes h = \gamma'' \otimes 5h$ is contained in $(P \times P)_* (\text{CH}_0(F) \otimes \text{CH}_1(F))$. Thus $\gamma \otimes h - \deg(\gamma) \mathbf{l} \otimes h$ can be absorbed into $(P \times P)_* \theta$.

When $d = 4$, it is known that $q_* p^* : H^6(F, \mathbb{Z}) \rightarrow H^4(X, \mathbb{Z})$ is an isomorphism. If we assume that the integral Hodge conjecture holds for 1-cycles on F , then the term Σ can always be absorbed into θ .

Proposition 5.3. *Let X be a smooth cubic hypersurface of dimension $d = 3$ or 4 . Let Γ be a symmetric cycle on $X \times X$ such that $[\Delta_X] = [\Gamma]$ in $H^d(X \times X, \mathbb{Z})$.*

(i) *If $d = 3$, then*

$$\Delta_X - \Gamma = 0, \quad \text{in } \text{CH}_d(X \times X)/\text{alg.}$$

(ii) *If $d = 4$, then there exists a homologically trivial cycle $\sum_i a_i S_i \in \text{CH}_2(F)$ such that*

$$\Delta_X - \Gamma - \sum a_i P_i \times_{S_i} P_i = 0, \quad \text{in } \text{CH}_4(X \times X)/\text{alg.},$$

where $P_i = P|_{S_i}$.

Proof. Recall that $h \in \text{CH}^1(X)$ is the class of a hyperplane section. Let $l \subset X$ be a line. We note that, by Totaro [23], all cohomology groups involved in the proof are torsion free and hence $H^*(-)$ should be understood to be $H^*(-, \mathbb{Z})$. By [29, Corollary 2.4], there exists a d -cycle Γ' on $X^{[2]}$ such that

$$\mu^* \Gamma' = \Delta_X - \Gamma$$

as algebraic cycles, where $\mu \in \text{CH}_{2d}((X \times X) \times X^{[2]})$ is the correspondence defined by the closure of the graph of the rational map $X^2 \dashrightarrow X^{[2]}$. As is explained in the proof of [29, Proposition 2.6], we can require that

$$[\Gamma'] = 0, \quad \text{in } H^{2d}(X^{[2]}).$$

If $d = 3$, then Lemma 5.4 (i) applies. If $d = 4$, then by Lemma 5.4 (ii) we can find homologically trivial $\sum_i a_i S_i \in \text{CH}_2(F)$, such that $\Delta_X - \Gamma - \sum_i a_i P_i \times_{S_i} P_i$ is algebraically trivial. \square

Lemma 5.4. *Let $\Gamma' \in \text{CH}^{2d}(X^{[2]})_{\text{hom}}$.*

(i) *If $d = 3$, then Γ' is algebraically trivial.*

(ii) *If $d = 4$, then there exist surfaces $S_i \subset F$ and integers a_i such that $\sum a_i [S_i] = 0$ in $H^4(F, \mathbb{Z})$ and*

$$\mu^* \Gamma' = \sum a_i P_i \times_{S_i} P_i, \quad \text{in } \text{CH}^d(X \times X)/\text{alg.},$$

where P_i is the universal line P restricted to S_i .

Proof. This is a consequence of the explicit resolution $\tilde{\Phi}$ of the birational map Φ between $X^{[2]}$ and $P_X := \mathbb{P}(\mathcal{T}_{\mathbb{P}^{d+1}}|_X)$. See the previous section and [29, Proposition 2.9]. Recall that we have the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{\pi_1} & P_1 \\ & \searrow j & \downarrow i_1 \\ & \widetilde{X^{[2]}} & \xrightarrow{\tilde{\Phi}} P_X \\ \pi_2 \downarrow & & \downarrow \tau \\ P_2 & \xrightarrow{i_2} & X^{[2]} \end{array}$$

Here $P_2 = P^{[2]}/F$ is the relative Hilbert scheme of two points on the universal line P/F and $P_1 = P$. The morphism τ is the blow up of $X^{[2]}$ along P_2 and $\tilde{\Phi}$ is the blow up of P_X along P_1 . The two blow up morphisms share the same exceptional divisor

$$E = P_1 \times_F P_2.$$

Note that $\eta_1 : P_1 \rightarrow F$ is a \mathbb{P}^1 -bundle over F and $\eta_2 : P_2 \rightarrow F$ is a \mathbb{P}^2 -bundle over F . Let $\xi_i \in \mathrm{CH}^1(P_i)$, $i = 1, 2$, be the first Chern classes of the relative $\mathcal{O}(1)$ -bundles. By abuse of notation, we still use ξ_i to denote its pull back to E . By the blow up formula, we know that

$$\mathrm{CH}_d(\widetilde{X^{[2]}}) = \tilde{\Phi}^* \mathrm{CH}_d(P_X) \oplus j_* \pi_1^* \mathrm{CH}_{d-2}(P_1) \oplus j_* (\xi_2 \cdot \pi_1^* \mathrm{CH}_{d-1}(P_1)).$$

Thus

$$(14) \quad \tau^* \Gamma' = \tilde{\Phi}^* \Gamma'_0 + j_* \pi_1^* \Gamma'_1 + j_* (\xi_2 \pi_1^* \Gamma'_2)$$

where

$$\Gamma'_0 \in \mathrm{CH}_d(P_X)_{\mathrm{hom}}, \quad \Gamma'_1 \in \mathrm{CH}_{d-2}(P_1)_{\mathrm{hom}}, \quad \Gamma'_2 \in \mathrm{CH}_{d-1}(P_1)_{\mathrm{hom}}.$$

By the projective bundle formula, we have

$$\begin{aligned} \Gamma'_1 &= \eta_1^* \Gamma'_{1,0} + \xi_1 \cdot \eta_1^* \Gamma'_{1,1}, & \Gamma'_{1,0} &\in \mathrm{CH}_{d-3}(F)_{\mathrm{hom}} \text{ and } \Gamma'_{1,1} \in \mathrm{CH}_{d-2}(F)_{\mathrm{hom}}; \\ \Gamma'_2 &= \eta_1^* \Gamma'_{2,0} + \xi_1 \cdot \eta_1^* \Gamma'_{2,1}, & \Gamma'_{2,0} &\in \mathrm{CH}_{d-2}(F)_{\mathrm{hom}} \text{ and } \Gamma'_{2,1} \in \mathrm{CH}_{d-1}(F)_{\mathrm{hom}}. \end{aligned}$$

Note that

$$\begin{aligned} \tau_* j_* (\pi_1^* \eta_1^* \Gamma'_{1,0}) &= i_{2,*} \pi_{2,*} (\pi_1^* \eta_1^* \Gamma'_{1,0}) = 0, \\ \tau_* j_* (\xi_2 \cdot \pi_1^* \eta_1^* \Gamma'_{2,0}) &= i_{2,*} \pi_{2,*} (\xi_2 \cdot \pi_1^* \eta_1^* \Gamma'_{2,0}) = 0. \end{aligned}$$

Applying σ_* to equation (14), we get

$$\begin{aligned} \Gamma' &= \Phi^* \Gamma'_0 + i_{2,*} \pi_{2,*} \pi_1^* (\xi_1 \cdot \eta_1^* \Gamma'_{1,1}) + i_{2,*} \pi_{2,*} (\xi_2 \cdot \pi_1^* (\xi_1 \cdot \eta_1^* \Gamma'_{2,1})) \\ &= \Phi^* \Gamma'_0 + i_{2,*} \eta_2^* \Gamma'_{1,1} + i_{2,*} (\xi_2 \cdot \eta_2^* \Gamma'_{2,1}). \end{aligned}$$

Now we follow the argument in [29, p. 11]. The first fact is that the algebraic equivalence is the same as the homological equivalence on P_X . Thus we see that Γ'_0 is algebraically equivalent to zero. When $d = 3$ or 4 , the cycle $\Gamma'_{2,1}$ is either of codimension 0 or of codimension 1. Thus we always have that $\Gamma'_{2,1}$ is algebraically equivalent to zero. When $d = 3$ we have $\dim F = 2$ and $\Gamma'_{1,1} \in \mathrm{CH}^1(F)_{\mathrm{hom}}$. So $\Gamma'_{1,1}$ is also algebraically equivalent to zero in this case. Statement (1) follows.

Assume $d = 4$ and hence $\dim F = 4$. Thus $\Gamma'_{1,1} \in \mathrm{CH}_2(F)_{\mathrm{hom}}$. We can write

$$\Gamma'_{1,1} = \sum_i a_i S_i$$

where $S_i \subset F$ are surfaces. Then an explicit computation gives

$$\mu^* (i_{2,*} \eta_2^* S_i) = P_i \times_{S_i} P_i$$

as cycles. Hence the lemma is proved. \square

5.2. Algebraicity of the Beauville–Bogomolov form. Let X be a smooth cubic fourfold and let F be its variety of lines. It is known that F is a hyperkähler variety. By Beauville–Donagi [4], we know that $\alpha \mapsto \hat{\alpha} := p_* q^* \alpha$ gives an isomorphism between $H^4(X, \mathbb{Z})$ and $H^2(F, \mathbb{Z})$ and the Beauville–Bogomolov bilinear form on $H^2(F, \mathbb{Z})$ is given by

$$\mathfrak{B}(\hat{\alpha}, \hat{\beta}) = \langle \alpha, h^2 \rangle_X \langle \beta, h^2 \rangle_X - \langle \alpha, \beta \rangle_X,$$

for all $\alpha, \beta \in H^4(X, \mathbb{Z})$. Let α_i , $i = 1, \dots, 23$, be an integral basis of $H^4(X, \mathbb{Z})$. Then $\{\hat{\alpha}_i\}$ form an integral basis of $H^2(F, \mathbb{Z})$ and let $\{\hat{\alpha}_i^\vee\}$ be the dual basis of $H^6(F, \mathbb{Z})$. Then the Beauville–Bogomolov form corresponds to the canonical integral Hodge class

$$q_{\mathfrak{B}} = \sum_{i,j=1}^{23} b_{ij} \hat{\alpha}_i^\vee \otimes \hat{\alpha}_j^\vee \in H^{12}(F \times F, \mathbb{Z}),$$

Where $b_{ij} = \mathfrak{B}(\hat{\alpha}_i, \hat{\alpha}_j)$.

Proposition 5.5. *Let X be a smooth cubic fourfold and F be its variety of lines as above. If $q_{\mathfrak{B}} \in H^{12}(F \times F, \mathbb{Z})$ is algebraic then X is universally CH_0 -trivial. The converse is true if the integral Hodge conjecture holds for 1-cycles on F (e.g. if $\mathrm{Hdg}^4(X)$ is generated by h^2).*

Proof. Assume that $q_{\mathfrak{B}}$ is algebraic. Since $[\Delta_X] + (P \times P)_* q_{\mathfrak{B}}$ pairs to zero with $\alpha \otimes \beta$ for all $\alpha, \beta \in H^4(X, \mathbb{Z})_{\text{prim}}$, we know that $[\Delta_X] + (P \times P)_* q_{\mathfrak{B}}$ is decomposable. Thus X admits a cohomological decomposition of the diagonal, which implies universal CH_0 -triviality by Voisin [29]. Assume that $\text{CH}_0(X)$ is universally trivial. Then we get the cycle $\theta \in \text{CH}_2(F \times F)$ as in Theorem 5.1. The Hodge class $q_{\mathfrak{B}} + [\theta]$ is decomposable and hence algebraic by the assumption that the integral Hodge conjecture holds for 1-cycles on F . \square

5.3. The minimal class on the intermediate Jacobian of a cubic threefold. Let X be a smooth cubic threefold and let F be the surface of lines on X . Let $J^3(X)$ be the intermediate Jacobian of X . It is known that the Abel–Jacobi map

$$\phi : F \rightarrow J^3(X)$$

induces an isomorphism

$$\phi^* : H^1(J^3(X), \mathbb{Z}) \longrightarrow H^1(F, \mathbb{Z}).$$

There is a natural identification $H^1(J^3(X), \mathbb{Z}) \cong H^3(X, \mathbb{Z})$, under which we have

$$(15) \quad \phi^* \alpha = \hat{\alpha} = p_* q^* \alpha,$$

for all $\alpha \in H^1(J^3(X), \mathbb{Z}) = H^3(X, \mathbb{Z})$. By taking the difference or the sum, we have the following morphisms

$$\begin{aligned} \phi_+ : F \times F &\longrightarrow J^3(X), & (u, v) &\mapsto \phi(u) + \phi(v), \\ \phi_- : F \times F &\longrightarrow J^3(X), & (u, v) &\mapsto \phi(u) - \phi(v). \end{aligned}$$

By [6, §13], the image of ϕ_- is a theta divisor of $J^3(X)$ and ϕ_- has degree 6 onto its image. Similarly, we also know that the image of ϕ_+ is a divisor D_+ of cohomological class 3Θ and ϕ_+ has degree 2 onto its image.

Proposition 5.6. *If $\text{CH}_0(X)$ is universally trivial, then the following holds.*

- (i) *The minimal class of $J^3(X)$ is algebraic and supported on the divisor $D_+ \subset J^3(X)$ of cohomology class 3Θ .*
- (ii) *Twice of the minimal class is represented by a symmetric 1-cycle supported on a theta divisor of $J^3(X)$.*

Proof. The morphism ϕ_+ factors as

$$F \times F \xrightarrow{(\phi, \phi)} J^3(X) \times J^3(X) \xrightarrow{\mu_+} J^3(X)$$

where $\mu_+(x, y) = x + y$ is the summation morphism. Thus for any $\alpha \in H^1(J^3(X), \mathbb{Z})$, we have

$$(\phi_+)^* \alpha = (\phi, \phi)^* (\mu_+)^* \alpha = (\phi, \phi)^* (\alpha \otimes 1 + 1 \otimes \alpha) = \phi^* \alpha \otimes 1 + 1 \otimes \phi^* \alpha.$$

Similarly, we also have $(\phi_-)^* \alpha = \phi^* \alpha \otimes 1 - 1 \otimes \phi^* \alpha$. Let $\theta \in \text{CH}_1(F \times F)$ be the symmetric cycle as in Theorem 5.1. Then for all $\alpha, \beta \in H^1(J^3(X), \mathbb{Z})$ we have

$$\begin{aligned} (\phi_+)_* [\theta] \cup \alpha \cup \beta &= [\theta] \cup (\phi_+)^* \alpha \cup (\phi_+)^* \beta \\ &= [\theta] \cup (\phi^* \alpha \otimes 1 + 1 \otimes \phi^* \alpha) \cup (\phi^* \beta \otimes 1 + 1 \otimes \phi^* \beta) \\ &= [\theta] \cup ((\hat{\alpha} \cup \hat{\beta}) \otimes 1 + \hat{\alpha} \otimes \hat{\beta} - \hat{\beta} \otimes \hat{\alpha} + 1 \otimes (\hat{\alpha} \cup \hat{\beta})) \\ &= 2\phi_* [\theta_1] \cup \alpha \cup \beta + 2\langle \alpha, \beta \rangle_X, \end{aligned}$$

where $\theta_1 = (pr_1)_* \theta \in \text{CH}_1(F)$. The same computation shows that

$$(\phi_+)_* [\theta_1 \otimes \mathfrak{o}] \cup \alpha \cup \beta = (\phi_+)_* [\mathfrak{o} \otimes \theta_1] \cup \alpha \cup \beta = \phi_* [\theta_1] \cup \alpha \cup \beta,$$

where $\mathfrak{o} \in J^3(X)$ is the zero element. Take $\tilde{\theta} = \theta - \theta_1 \otimes \mathfrak{o} - \mathfrak{o} \otimes \theta_1$, then

$$(\phi_+)_* [\tilde{\theta}] \cup \alpha \cup \beta = 2\langle \alpha, \beta \rangle_X.$$

Since $\tilde{\theta}$ is again symmetric, we know that $(\phi_+)_*\tilde{\theta} = 2\eta$ for some $\eta \in \text{CH}_1(J^3(X))$. Thus $[\eta] \cup \alpha \cup \beta = \langle \alpha, \beta \rangle_X$ and hence $-\eta$ is the minimal class on $J^3(X)$. It is also clear that η is supported on D_+ . We carry out the same computation for ϕ_- and see that

$$(\phi_-)_*[\tilde{\theta}] \cup \alpha \cup \beta = -2\langle \alpha, \beta \rangle_X.$$

Thus the cohomology class of $(\phi_-)_*\tilde{\theta}$ is twice the minimal class. Furthermore it is a symmetric (with respect to multiplication by -1 on $J^3(X)$) 1-cycle supported on the image of ϕ_- which is a theta divisor. \square

5.4. Cubic fivefolds. Let $X \subset \mathbb{P}^6$ be a cubic fivefold and let F be its variety of lines. Let $J^5(X)$ be the intermediate Jacobian of X , which happens to be a principally polarized abelian variety.

Proposition 5.7. *If both $\text{CH}_0(X)$ and $\text{CH}_0(F)$ are universally trivial, then there exist curves finitely many C_i together with a splitting surjective homomorphism $\bigoplus_i J(C_i) \rightarrow J^5(X)$.*

Proof. Since $\text{CH}_0(F) \rightarrow \text{CH}_1(X)$ is universally surjective, we know that $\text{CH}_1(X)$ is universally trivial. The universal triviality of both $\text{CH}_0(X)$ and $\text{CH}_1(X)$ implies the existence of curves C_i , correspondences $\Gamma_i \in \text{CH}^3(Z_i \times X)$ and integers n_i such that

$$(16) \quad \sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{C_i} = \langle \alpha, \beta \rangle_X,$$

for all $\alpha, \beta \in F^5 H^5(X, \mathbb{Z}) = H^5(X, \mathbb{Z})$, where $\sigma_i : C_i \rightarrow C_i$ is either the identity map or an involution. Each cycle Γ_i defines the associated Abel–Jacobi map

$$\phi_i : J(C_i) \rightarrow J^5(X).$$

Combining them together we have the surjective homomorphism

$$\phi : \bigoplus_i J(C_i) \rightarrow J^5(X).$$

Then the equation (16) implies that ϕ has a section given by $\sum \sigma_i^\vee \circ \phi_i^\vee$. \square

REFERENCES

- [1] N. Addington, B. Hassett, Y. Tschinkel and A. Várilly-Alvarado, *Cubic fourfolds fibered in sextic del Pezzo surfaces*, preprint 2016, arXiv:1606.05321.
- [2] A.B. Altman and S.L. Kleiman, *Foundations of the theory of Fano schemes*, Compos. Math. **34** (1), 1977, 3–47.
- [3] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95.
- [4] A. Beauville and Donagi, *La variété des droites d’une hypersurface cubique de dimension 4*. C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.
- [5] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. of Math. **105** (1983), 1235–1253.
- [6] C.H. Clemens and P.A. Griffiths, *Intermediate Jacobian of cubic fourfolds*, Ann. Math. **95** No. 2, 1972, 281–356.
- [7] J.-L. Colliot-Thélène and A. Pirutka, *Hypersurfaces quartiques de dimension 3: nonrationalité stable*, Annales Scientifiques de l’ENS **49**, fascicule 2 (2016), 371–397.
- [8] S. Galkin and E. Shinder, *The Fano variety of lines and rationality problem for a cubic hypersurface*, preprint, 2014.
- [9] B. Hassett, *Special cubic fourfolds*, Compos. Math. **120**, 1–23, 2000.
- [10] B. Hassett, *Some rational cubic fourfolds*, J. Alg. Geom. **8** (1999), no. 1, 103–114.
- [11] B. Hassett, A. Kresch and Y. Tschinkel, *Stable rationality and conic bundles*, Math. Ann. **365** (2016), no. 3–4, 1201–1217.
- [12] B. Hassett, A. Pirutka and Y. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, preprint 2016, arXiv: 1603.09262.
- [13] B. Hassett, A. Pirutka and Y. Tschinkel, *A very general quartic double fourfold is not stably rational*, preprint 2016, arXiv: 1605.03220.
- [14] B. Hassett and Y. Tschinkel, *On stable rationality of Fano threefolds and del Pezzo fibrations*, J. Reine Angew. Math., to appear.
- [15] V. A. Iskovskikh and Yu. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. (N.S.) **86**(128) (1971), 140–166.
- [16] A. Kuznetsov, *Derived categories of cubic fourfolds*, in “Cohomological and geometric approaches to rationality problems”, 219–243, Progr. Math. 282, Birkhäuser Boston, Inc., Boston, MA, 2010.
- [17] T. Okada, *Stable rationality of cyclic covers of projective spaces*, preprint 2016, arXiv: 1604.08417.
- [18] M. Shen, *On relations among 1-cycles on cubic hypersurfaces*, J. Alg. Geom. **23** (2014), 539–569.
- [19] M. Shen, *Prym–Tjurin construction on cubic hypersurfaces*, Doc. Math. **19** (2014), 867–903.

- [20] M. Shen, *Hyperkähler manifolds of Jacobian type*, J. reine Angew. Math., to appear.
- [21] I. Shimada, *On the cylinder isomorphism associated to the family of lines on a hypersurface*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **37** (1990), 703–719.
- [22] B. Totaro, *Hypersurfaces that are not stably rational*, J. Amer. Math. Soc., to appear.
- [23] B. Totaro, *The integral cohomology of the Hilbert scheme of two points*, preprint 2015.
- [24] V. Voevodsky, *A nilpotence theorem for cycles algebraically equivalent to zero*, Internat. Math. Res. Notices, No. 4 (1995), 187–198.
- [25] C. Voisin, *Remarks on zero-cycles of self-products of varieties*, in “Moduli of vector bundles” (Proceedings of the Taniguchi congress on vector bundles), Maruyama Ed., Decker (1994), 265–285.
- [26] C. Voisin, *Some aspects of the Hodge conjecture*, Jpn. J. Math. **2** (2007), no. 2, 261–296.
- [27] C. Voisin, *Abel–Jacobi map, integral Hodge classes and decomposition of the diagonal*, J. Alg. Geom. **22** (2013), no. 1, 141–174.
- [28] C. Voisin, *Unirational threefolds with no universal codimension 2 cycle*, Invent. Math. **201** (2015), no. 1, 207–237.
- [29] C. Voisin, *On the universal trivial CH_0 of cubic hypersurfaces*, JEMS, to appear.

KdV INSTITUTE FOR MATHEMATICS, UNIVERSITY OF AMSTERDAM, P.O.Box 94248, 1090 GE AMSTERDAM, NETHERLANDS
E-mail address: M.Shen@uva.nl